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Quantum Harmonic Oscillator with Time-Dependent Mass and Frequency

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The quantum harmonic oscillator with time-dependent mass and frequency is analyzed by using the canonical transformation method. The varying mass and frequency of the system are reduced to constant mass and frequency, and the corresponding eigenvalues and eigenvectors are derived. The exact time-dependent coherent state of the harmonic oscillator is constructed and shown to be equivalent to the squeezed state. Damped harmonic oscillators with different frictions and forced time-dependent harmonic oscillators are also discussed.

1. INTRODUCTION

The study of the time-dependent harmonic oscillator with time-dependent frequency (Hartley and Ray, 1982a,b; Pedrosa, 1987a,b), or with time-dependent mass (Leach, 1983; Colegrave and Abdalla, 1981a,b, 1982, 1983a,b), or both simultaneously (Abdalla, 1986; Jannussis and Bartzis, 1988a,b; Lo, 1990, 1992; Baseia and De Brito, 1992; Dantas *et al.*, 1992) has attracted considerable interest in different areas of physics, such as plasma physics, gravitation, and quantum optics. Colegrave and Abdalla (1983a,b) studied the harmonic oscillator with constant frequency and time-dependent mass in order to describe the electromagnetic field in a Fabry–Perot cavity. Hartley and Ray (1982a) and Pedrosa (1987a) treated the time-dependent harmonic oscillator with varying frequency and constant mass for studying the coherent state. Dantas *et al.* (1992) solved the harmonic oscillator under the action of a particular time-dependent perturbative potential. Lo (1992) studied the time evolution of a charged oscillator with time-dependent mass

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and frequency in a time-dependent electromagnetic field and obtained the eigenfunctions and eigenstates at any time t. Baseia and De Brito (1992) examined the generation of squeezing for a harmonic oscillator when a sudden change of mass takes place.

Many methods have been presented to solve the problems of time-dependent harmonic oscillators, but in essence there are two main approaches: the first, based on Lewis and Riesenfeld (1969), uses time-dependent invariants, and the second uses time-dependent canonical transformations (Pedrosa, 1987b). In this paper the canonical transformation method is used to solve the harmonic oscillator with time-dependent mass and frequency described by the Hamiltonian

$$\hat{H} = \frac{\hat{p}^2}{2m(t)} + \frac{1}{2} m(t)\omega^2(t)\hat{q}^2 \tag{1.1}$$

This Hamiltonian can be canonically transformed to a new Hamiltonian with time-independent mass and frequency which has time-dependent eigenvalues and eigenvectors.

The canonical transformation method will be introduced in Section 2. The cases for $k = 0$ and $k \neq 0$ are treated in Sections 3 and 4, respectively. When $k = 0$, the exact solutions of position and momentum operators are obtained, and the time-dependent annihilation and creation operators are shown to be equivalent to the squeezed operators. When $k \neq 0$, the time-dependent state is constructed and it is equivalent to the squeezed state. Damped harmonic oscillators with different frictions and forced time-dependent harmonic oscillators are discussed in Sections 5 and 6. Finally, a reasonable physical interpretation is given in Section 7.

2. LINEAR CANONICAL TRANSFORMATION

In the Heisenberg picture, the equations of motion for the Hamiltonian (1.1) are

$$\frac{d\hat{q}}{dt} = \frac{1}{i\hbar}[\hat{q}, \hat{H}] = \frac{\hat{p}}{m(t)} \tag{2.1}$$

$$\frac{d\hat{p}}{dt} = \frac{1}{i\hbar}[\hat{p}, \hat{H}] = -m(t)\omega^2(t)\hat{q} \tag{2.2}$$

The Hamiltonian (1.1) can be canonically transformed by using the transformation equations $P = \partial F/\partial q$ and $Q = \partial F/\partial p$. We have

$$\hat{Q}(t) = \frac{1}{A(t)}\hat{q} \tag{2.3}$$

$$\hat{P}(t) = A(t)\hat{p} + \Phi(t)\hat{q} \tag{2.4}$$

We introduce the generating function

$$F(\hat{q}, \hat{P}, t) = \frac{1}{2A(t)}(\hat{q}\hat{P} + \hat{P}\hat{q}) - \frac{\Phi(t)}{2A(t)}\hat{q}^2 \tag{2.5}$$

where $A(t)$ and $\Phi(t)$ are any functions of time. From equations (2.1)–(2.4) we have

$$\frac{d\hat{Q}}{dt} = -\frac{[m(t)A'(t) + \Phi(t)]}{A^2(t)m(t)}\hat{q} + \frac{\hat{P}}{A^2(t)m(t)} \tag{2.6}$$

$$\frac{d\hat{P}}{dt} = \frac{1}{m(t)}[m(t)A'(t) + \Phi(t)]\hat{p} - [m(t)A(t)\omega^2(t) - \Phi'(t)]\hat{q} \tag{2.7}$$

Here the prime denotes d/dt. In order to solve the problem and to simplify it, we let

$$m(t)A'(t) + \Phi(t) = 0 \tag{2.8}$$

$$A(t)\omega^2(t)m(t) - \Phi'(t) = \frac{k^2}{m(t)A^3(t)} \tag{2.9}$$

with the initial condition

$$A(0) = 0, \qquad \Phi(0) = 0 \tag{2.10}$$

From equations (2.6) and (2.7) we have

$$\frac{d\hat{Q}}{dt} = \frac{\hat{P}}{m(t)A^2(t)} \tag{2.11}$$

$$\frac{d\hat{P}}{dt} = -\frac{k^2\hat{Q}}{m(t)A^2(t)} \tag{2.12}$$

From equations (2.8) and (2.9), we get the auxiliary equation

$$A''(t) + \gamma(t)A'(t) + \omega^2(t)A(t) = \frac{k^2}{m^2(t)A^3(t)} \tag{2.13}$$

with the initial condition

$$A(0) = 1, \qquad A'(0) = 0$$

where

$$\gamma(t) = \frac{d}{dt} \ln m(t)$$

The function $\Phi(t)$ can be expressed as

$$\Phi(t) = -m(t)A'(t) = \int_0^t \left[m(t')A(t')\omega^2(t') - \frac{k^2}{m(t')A^3(t')} \right] dt' \qquad (2.14)$$

Note that $A(t)$, $\Phi(t)$, and k are all real, and let $k > 0$. As a result, the time-dependent system (1.1) becomes a new system with the transformed Hamiltonian

$$\hat{\bar{H}} = \frac{1}{m(t)A^2(t)} \left(\frac{\hat{P}^2}{2} + \frac{k^2\hat{Q}^2}{2} \right) \qquad (2.15)$$

From equations (2.3), (2.4), and (2.8) we have

$$\hat{Q}(t) = \frac{\hat{q}}{A(t)} \qquad (2.16)$$

$$\hat{P}(t) = A(t)\hat{p} - m(t)A'(t)\hat{q} \qquad (2.17)$$

In the Schrödinger picture, we have

$$\hat{\bar{H}}\psi(\hat{Q}, t) = i\hbar\, \partial\psi/\partial t \qquad (2.18)$$

or equivalently

$$\hat{\bar{H}}_1\psi(\hat{Q}, t) = m(t)A^2(t)i\hbar\, \partial\psi/\partial t \qquad (2.19)$$

where

$$\hat{\bar{H}}_1(\hat{Q}) = \frac{\hat{P}^2}{2} + \frac{k^2\hat{Q}^2}{2} \qquad (2.20)$$

The solution of equation (2.18) or equation (2.19) can be written as

$$\begin{aligned} \psi(\hat{Q}, t) &= \sum_n C_n\psi_n(\hat{Q}, t) \\ &= \sum_n C_n \exp\left[-i\hat{\bar{H}}_1(\hat{Q})/\hbar \int_0^t \frac{1}{m(t')A^2(t')}\, dt' \right] \Phi_n(\hat{Q}) \end{aligned} \qquad (2.21)$$

where $\Phi_n(Q)$ is the solution of the eigenvalue equation

$$\hat{H}_1(\hat{Q})\Phi_n(\hat{Q}) = E_n\Phi_n(\hat{Q}) \tag{2.22}$$

with

$$\langle\Phi_n/\Phi_n\rangle = 1$$

3. THE CASE OF $k = 0$

For the harmonic oscillator with time-dependent mass and frequency, we can obtain the transformed equations of motion from equations (2.11) and (2.12),

$$d\hat{Q}/dt = \hat{P}/m(t)A^2(t) \tag{3.1}$$

$$d\hat{P}/dt = 0 \tag{3.2}$$

From equations (3.1) and (3.2), we have

$$\hat{P}(t) = \rho, \qquad \hat{Q}(t) = \int_0^t \frac{\rho}{m(t')A^2(t')}\,dt' + \beta \tag{3.3}$$

where ρ and β are constants. From equations (2.16) and (2.17), we have

$$\hat{q}(t) = \hat{q}_0 A(t) + \hat{p}_0 A(t)\int_0^t \frac{1}{A^2(t)m(t)}\,dt \tag{3.4}$$

$$\hat{p}(t) = \left[\frac{1}{A(t)} + m(t)A'(t)\int_0^t \frac{1}{A^2(t)m(t)}\,dt\right]\hat{p}_0 + m(t)A'(t)\hat{q}_0 \tag{3.5}$$

where

$$\hat{q}_0 = \hat{q}(t=0) = \hat{Q}(0) = \beta, \qquad \hat{p}_0 = \hat{p}(t=0) = \hat{P}(0) = \rho$$

Construct the annihilation and creation operators

$$\hat{a}(t) = \frac{1}{(2\hbar)^{1/2}}\left\{[m(t)\omega(t)]^{1/2}\hat{q}(t) + i\,\frac{\hat{p}(t)}{[m(t)\omega(t)]^{1/2}}\right\} \tag{3.6}$$

$$\hat{a}^+(t) = \frac{1}{(2\hbar)^{1/2}}\left\{[m(t)\omega(t)]^{1/2}\hat{q}(t) - i\,\frac{\hat{p}(t)}{[m(t)\omega(t)]^{1/2}}\right\} \tag{3.7}$$

which satisfy the commutation relation $[\hat{a}, \hat{a}^{+}] = 1$. From equations (3.4)–(3.7), we have

$$\hat{a}(t) = \mu(t)\hat{a}(0) + \gamma(t)\hat{a}^{+}(0) \tag{3.8}$$

$$\hat{a}^{+}(t) = \mu^{*}(t)\hat{a}^{+}(0) + \gamma^{*}(t)\hat{a}(0) \tag{3.9}$$

with

$$\mu(t) = \frac{1}{2}\left(\left\{\left[\frac{m(t)\omega(t)}{m\omega_0}\right]^{1/2} A(t) + \frac{1}{A(t)}\left[\frac{m\omega_0}{m(t)\omega(t)}\right]^{1/2} + A'(t)M(t)\left[\frac{m\omega_0 m(t)}{\omega(t)}\right]^{1/2}\right\} + i\left\{A'(t)\left[\frac{m(t)}{m\omega(t)\omega_0}\right]^{1/2} - A(t)M(t)[m\omega(t)m(t)\omega_0]^{1/2}\right\}\right) \tag{3.10}$$

$$\gamma(t) = \frac{1}{2}\left(\left\{\left[\frac{m(t)\omega(t)}{m\omega_0}\right]^{1/2} A(t) - \frac{1}{A(t)}\left[\frac{m\omega_0}{m(t)\omega(t)}\right]^{1/2} - A'(t)M(t)\left[\frac{m\omega_0 m(t)}{\omega(t)}\right]^{1/2}\right\} + i\left\{A'(t)\left[\frac{m(t)}{m\omega(t)\omega_0}\right]^{1/2} + A(t)M(t)[m\omega(t)m(t)\omega_0]^{1/2}\right\}\right) \tag{3.11}$$

where

$$\omega_0 = \omega(0), \qquad m = m(0), \qquad M(t) = \int_0^t \frac{1}{m(t')A^2(t')}\, dt'$$

The functions $\mu(t)$ and $\gamma(t)$ satisfy the relation

$$|\mu|^2 - |\gamma|^2 = 1 \tag{3.12}$$

Therefore, the operators (3.6) and (3.7) are the squeezed operators. According to Yuan (1976), the uncertainty in $\hat{p}$ and $\hat{q}$ for the squeezed state is given by

$$\langle\Delta q\rangle\langle\Delta p\rangle = (\hbar/2)|\mu - \gamma|\cdot|\mu + \gamma| \tag{3.13}$$

From the above argument we see that if the initial state is a coherent state, the corresponding time-evolved state is a squeezed state.

4. THE CASE OF $k \neq 0$

From equation (2.20) we have

$$\hat{\bar{H}}_1 = \left[\hat{a}^+(t)\hat{a}(t) + \frac{1}{2}\right]\hbar k \tag{4.1}$$

where

$$\hat{a}(t) = \frac{1}{(2\hbar)^{1/2}}\left[(k\hat{Q})^{1/2} + i\frac{\hat{P}}{k^{1/2}}\right] \tag{4.2}$$

$$\hat{a}^+(t) = \frac{1}{(2\hbar)^{1/2}}\left[(k\hat{Q})^{1/2} - i\frac{\hat{P}}{k^{1/2}}\right] \tag{4.3}$$

The annihilation and creation operators $\hat{a}(t)$ and $\hat{a}^+(t)$ satisfy the commutation relation

$$[\hat{a}(t), \hat{a}^+(t)] = 1 \tag{4.4}$$

Thus the eigenfunction of $\hat{\bar{H}}_1$ is the number state, i.e.,

$$\Phi_n(\hat{Q}) = |n\rangle \tag{4.5}$$
$$\hat{\bar{H}}_1|n\rangle = \left(n + \frac{1}{2}\right)\hbar k|n\rangle$$

According to the procedure in Section 2, the time-evolved number state of the time-dependent system is

$$|n, t\rangle = \exp\left[-i\left(n + \frac{1}{2}\right)k\int_0^t \frac{dt'}{m(t')A^2(t')}\right]|n\rangle \tag{4.6}$$

and the general solution is

$$|\psi(t)\rangle = \sum_n C_n|n, t\rangle$$
$$= \sum_n C_n \exp\left[-i\left(n + \frac{1}{2}\right)k\int_0^t \frac{dt'}{m(t')A^2(t')}\right]|n\rangle \tag{4.7}$$

When $t = 0$, the Hamiltonian (1.1) can be written as

$$\hat{H}_0 = \left[\hat{a}^+(0)\hat{a}(0) + \frac{1}{2}\right]\hbar\omega_0 \tag{4.8}$$

where

$$\hat{a}(0) = \frac{1}{(2\hbar)^{1/2}} \left(\omega_0^{1/2} \hat{q} + i \frac{\hat{p}}{\omega_0^2} \right) \tag{4.9}$$

$$\hat{a}^+(0) = \frac{1}{(2\hbar)^{1/2}} \left(\omega_0^{1/2} \hat{q} - i \frac{\hat{p}}{\omega_0^{1/2}} \right) \tag{4.10}$$

The coherent state at $t = 0$ is

$$|a, 0\rangle = \exp\left(-\frac{|a|^2}{2} \right) \sum_n \frac{a^n}{(n!)^{1/2}} |n\rangle \tag{4.11}$$

Then we obtain the evolved coherent state

$$|a, t\rangle = \exp\left(-\frac{|a|^2}{2} \right) \sum_n \frac{a^n}{(n!)^{1/2}} |n, t\rangle \tag{4.12}$$

and equations (4.2) and (4.3) can be written as

$$\hat{a}(t) = \mu(t)\hat{a}(0) + \gamma(t)\hat{a}^+(0) \tag{4.13}$$

$$\hat{a}^+(t) = \mu^*(t)\hat{a}^+(0) + \gamma^*(t)\hat{a}(0) \tag{4.14}$$

with

$$\mu(t) = \frac{1}{2} \left\{ \frac{1}{A(t)} \left[\frac{k}{\omega(t)m(t)} \right]^{1/2} + A\,(t) \left[\frac{m\,(t)\;\omega\;(1)}{k} \right]^{1/2} + i\Phi(t) \frac{1}{[k\omega(t)m(t)]^{1/2}} \right\} \tag{4.15}$$

$$\gamma(t) = \frac{1}{2} \left\{ \frac{1}{A(t)} \left[\frac{k}{\omega(t)m(t)} \right]^{1/2} - A(t) \left[\frac{m(t)\omega(t)}{k} \right]^{1/2} + i\Phi(t) \frac{1}{[k\omega(t)m(t)]^{1/2}} \right\} \tag{4.16}$$

The functions $\mu(t)$ and $\gamma(t)$ satisfy the relation

$$|\mu|^2 - |\gamma|^2 = 1 \tag{4.17}$$

Therefore, the evolved coherent state is equivalent to the squeezed state. The uncertainty relation for the squeezed state is

$$\langle \Delta q \rangle \langle \Delta p \rangle = \left(-\frac{\hbar}{2}\right)\left[1 + \frac{A'^2(t)A^2(t)}{k^2}\right]^{1/2} \geq \frac{\hbar}{2}$$

These results indicate that the squeezed state can be generated from a coherent state by varying the mass and frequency of the oscillator.

5. DAMPED HARMONIC OSCILLATOR

We consider the case of damped harmonic oscillator, i.e.,

$$m(t) = m \cdot \exp(\gamma t), \qquad \omega(t) = \omega = \text{const}, \qquad k = 0$$

Then the auxiliary equation is

$$A''(t) + \gamma A'(t) + \omega^2 A(t) = 0 \tag{5.1}$$

with the initial condition

$$A(0) = 1, \qquad A'(0) = 0$$

The solution of equation (5.1) is

$$A(t) = \exp(-\gamma t/2)\left(\cosh \Omega t + \frac{\gamma}{2\Omega} \sinh \Omega t\right) \tag{5.2}$$

For strong friction $\Omega^2 = \gamma^2/4 - \omega^2 > 0$, we have

$$M(t) = \frac{(1/\Omega) \sinh \Omega t}{m[\cosh \Omega t + (\gamma/2\Omega) \sinh \Omega t]}$$

$$\hat{q}(t) = e^{-\gamma t/2}\left[\left(\cosh \Omega t + \frac{\gamma}{2\Omega} \sinh \Omega t\right)\hat{q}_0 + \frac{\hat{p}_0}{m\Omega} \sinh \Omega t\right] \tag{5.3}$$

$$\hat{p}(t) = e^{\gamma t/2}\left[\left(\cosh \Omega t - \frac{\gamma}{2\Omega} \sinh \Omega t\right)\hat{p}_0 - \frac{m\omega^2 \hat{q}_0}{\Omega} \sinh \Omega t\right] \tag{5.4}$$

Therefore the functions $\mu(t)$ and $\gamma(t)$ have the form

$$\mu(t) = \cosh \Omega t - i\left(\frac{\omega}{\Omega}\right) \sinh \Omega t$$

$$\gamma(t) = \frac{\gamma}{2\Omega} \sinh \Omega t$$

The uncertainty in $\hat{q}$ and $\hat{p}$ is given by

$$\langle\Delta q\rangle\langle\Delta p\rangle = \frac{\hbar}{2}\left(1 + \frac{\gamma^2\omega^2}{\Omega^4}\sinh^4\Omega t\right)^{1/2} \geq \frac{\hbar}{2} \tag{5.5}$$

For weak friction $\Omega^2 = \omega^2 - \gamma^2/4 > 0$, we have

$$\mu(t) = \cos\Omega t - i\left(\frac{\omega}{\Omega}\right)\sin\Omega t$$

$$\gamma(t) = \frac{\gamma}{2\Omega}\sin\Omega t$$

The uncertainty in $\hat{q}$ and $\hat{p}$ is given by

$$\langle\Delta q\rangle\langle\Delta p\rangle = \frac{\hbar}{2}\left(1 + \frac{\gamma^2\omega^2}{\Omega^4}\sin^4\Omega t\right)^{1/2} \geq \frac{\hbar}{2} \tag{5.6}$$

It takes the minimum value $\langle\Delta q\rangle\langle\Delta p\rangle = \hbar/2$ at $\Omega t = n\pi$ ($n = 0, 1, 2, \ldots$).

For the critical friction $\Omega \to 0$, we have

$$\mu(t) = 1 - i\omega t$$

$$\gamma(t) = \omega t$$

The uncertainty relation gives

$$\langle\Delta q\rangle\langle\Delta p\rangle = \frac{\hbar}{2}(1 + 4\omega^2t^2)^{1/2} \geq \frac{\hbar}{2}$$

The above result satisfies the property of the coherent state, and it is consistent with that of Jannussis and Bartzis (1988a,b).

6. FORCED TIME-DEPENDENT HARMONIC OSCILLATOR

We consider the forced harmonic oscillator with time-dependent mass and frequency, and the Hamiltonian is

$$\hat{H}_F = \frac{\hat{p}^2}{2m(t)} + \frac{1}{2}m(t)\omega^2(t)\hat{q}^2 + f(t)\hat{q} \tag{6.1}$$

where $f(t)$ is any force function of t. To simplify the treatment, we let $f(0) = 0$.

In the Heisenberg picture, the equations of motion are

$$\frac{d\hat{q}}{dt} = \frac{1}{i\hbar}[\hat{q}, \hat{H}_F] = \frac{\hat{p}}{m(t)} \tag{6.2}$$

$$\frac{d\hat{p}}{dt} = \frac{1}{i\hbar}[\hat{p}, \hat{H}_F] = -m(t)\omega^2(t)\hat{q} - f(t) \tag{6.3}$$

Following the procedure in Section 2, we make the canonical transformation

$$\hat{Q} = \frac{\hat{q} + c(t)}{A(t)} \tag{6.4}$$

$$\hat{P} = A(t)(\hat{p} + b(t)) - m(t)A'(t)(\hat{q} + c(t)) \tag{6.5}$$

where $b(t)$ and $c(t)$ are any functions of t, and $A(t)$ is the function satisfying the auxiliary equation (2.13).

Provided that $b(t)$ and $c(t)$ satisfy the relations

$$b(t) = m(t)c'(t) \tag{6.6}$$

$$b'(t) + m(t)\omega^2(t)c(t) = f(t) \tag{6.7}$$

with the initial condition

$$b(0) = 0, \qquad c(0) = 0$$

we obtain a new Hamiltonian of the form

$$\hat{\tilde{H}}_F = \frac{1}{m(t)A^2(t)}\left(\frac{\hat{P}^2}{2} + \frac{k^2\hat{Q}^2}{2}\right) \tag{6.8}$$

where

$$\hat{Q} = \frac{\hat{q} + c(t)}{A(t)} \tag{6.9}$$

$$\hat{P} = A(t)[\hat{p} + m(t)c'(t)] - m(t)A'(t)\hat{q} \tag{6.10}$$

The new equations of motion are

$$\frac{d\hat{Q}}{dt} = \frac{\hat{P}}{m(t)A^2(t)} \tag{6.11}$$

$$\frac{d\hat{P}}{dt} = -\frac{k^2\hat{Q}}{m(t)A^2(t)} \tag{6.12}$$

From Eqs. (6.6) and (6.7), we get another auxiliary equation

$$c^*(t) + \gamma(t)c'(t) + \omega^2(t)c(t) = \frac{f(t)}{m(t)} \tag{6.13}$$

with the initial condition

$$c'(0) = c(0) = 0$$

where

$$\gamma(t) = \frac{d}{dt}[\ln m(t)]$$

The function $b(t)$ can be expressed as

$$b(t) = m(t)c'(t) = \int_0^t [f(t') - m(t')\omega^2(t')c(t')]\, dt' \tag{6.14}$$

According to the procedure in Section 3, we can obtain in the case of $k = 0$

$$\hat{q}(t) = \hat{q}_0 A(t) + \hat{p}_0 A(t) \int_0^t \frac{1}{A^2(t')m(t')} dt' - c(t) \tag{6.15}$$

$$\hat{p}(t) = \left[\frac{1}{A(t)} + m(t)A'(t) \int_0^t \frac{1}{A^2(t')m(t')} dt'\right]\hat{p}_0 + m(t)A'(t)\hat{q}_0 - m(t)c'(t) \tag{6.16}$$

The annihilation and creation operators are also the squeezed operators, i.e.,

$$\hat{a}(t) = \mu(t)\hat{a}(0) + \gamma(t)\hat{a}^+(0) - \frac{1}{(2\hbar)^{1/2}}\left\{[m(t)\omega(t)]^{1/2}c(t) + i\frac{[m(t)]^{1/2}}{[\omega(t)]^{1/2}} c'(t)\right\} \tag{6.17}$$

$$\hat{a}^+(t) = \mu^*(t)\hat{a}^*(0) + \gamma^*(t)\hat{a} - \frac{1}{(2\hbar)^{1/2}}\left\{[m(t)\omega(t)]^{1/2}c(t) - i\frac{[m(t)]^{1/2}}{[\omega(t)]^{1/2}} c'(t)\right\} \tag{6.18}$$

where $\mu(t)$ and $\gamma(t)$ satisfy (3.10)–(3.12). The uncertainty in $\hat{p}$ and $\hat{q}$ gives

$$\langle\Delta q\rangle\langle\Delta p\rangle = \frac{\hbar}{2}|\mu + \gamma|\cdot|\mu - \gamma| \tag{6.19}$$

According to the procedure in Section 6, we obtain in the case of $k \neq 0$

$$|a, t\rangle = \exp\left(\frac{-|a|^2}{2}\right)\sum_n \frac{a^n}{(n!)^{1/2}}|n, t\rangle = \exp\left(\frac{-|a|^2}{2}\right)\sum_n \frac{a^n}{(n!)^{1/2}} \exp\left[-i\left(n + \frac{1}{2}\right)k \int_0^t \frac{dt'}{A^2(t')m(t')}\right]|n\rangle \tag{6.20}$$

This evolved coherent state is equivalent to that of the time-dependent harmonic oscillator. We can also prove that the evolved coherent state is just a squeezed state.

7. CONCLUSION

The time-dependent harmonic and forced time-dependent harmonic oscillators have been analyzed by the time-dependent linear canonical transformation method. The exact eigenvalues and eigenvectors of the harmonic oscillator with time-dependent mass and frequency were derived.

Leach (1977a,b, 1978; Gunther and Leach, 1977) used the linear canonical transformation to give a physical interpretation to the problem of finding an invariant and to solve some special time-dependent systems. But Leach only discussed the case of $k = 1$. In this paper, the case of $k = 0$ was also considered. The physical interpretation of this case is that the canonical coordinates (p, q) are transformed into new ones (P, Q) where the momentum operator P is invariant. By this transformation, the exact solutions of position and momentum operators can be obtained.

When $k \neq 0$, the exact coherent state is constructed by an effective method, which is different from that in Hartley and Ray (1982a). Our result is in good agreement with that of Dantas *et al.* (1992). The case of $k = 0$ has also been discussed. It was shown that the time-evolved state will be a squeezed state if the wave function of the time-dependent oscillator at $t = 0$ is a coherent state. Hence the squeezed state can be generated by externally changing the mass and frequency of the oscillator.

Finally, the damped harmonic oscillators with strong, weak, and critical frictions have been discussed. The result is consistent with that of Jannussis and Bartzis (1988a).

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